Incomplete information, three-valued logics and formal concept analysis

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In many situations we face, information is incomplete. In practice, it means that we cannot answer all questions due to our lack of knowledge, which is thus one source of uncertainty. The simplest representation of incomplete knowledge is a set of mutually exclusive values, one of which is the true one, also called a disjunctive set. In such a setting, possibility and necessity measures qualify the plausibility and the certainty of events; see [8] for a recent survey.

It contrasts with uncertainty due to randomness, which is a feature of observed phenomena, while the lack of knowledge is subjective, i.e., is agent-dependent. Of course, we may combine both sources of uncertainty and have limited information about a random phenomenon (e.g. see [5], for instance).

The lack of knowledge is generally present in propositional logic, if a set of formulas is viewed as pieces of information possessed by an agent. Uncertainty occurs when neither a proposition nor the converse proposition can be proved from this set of formulas. To handle incomplete knowledge in Boolean logic one has to move from the dichotomy *true/false* to what Yao calls three-way decision, i.e , *known to be true, known to be false and unknown.* Unfortunately, it is not possible to express this three-way reasoning in propositional logic, because we cannot write in the syntax that a proposition is unknown.

This talk first recalls the logic called MEL (minimal epistemic logic) which is the simplest formalism for reasoning about incomplete knowledge, where it is possible to express that some propositions are unknown to an agent [1]. It is at the same time a fragment of the modal logic KD (without modality nesting), and an all-or-nothing possibilistic logic, i.e., its semantics are in terms of disjunctive sets of interpretations viewed as possibility distributions, and representing epistemic states.

The truth of $\Box p$ expresses that $N(p) = 1$, that is, the certainty of p for the agent (following a principle first proposed a long time ago by Godo and colleagues to cast uncertainty calculi in a fuzzy logic setting; see [7] for another recent example). The language of MEL has for atoms modal formulas $\Box p$ where p is any proposition of a standard propositional logic. It is then a two-tiered propositional logic. MEL enables an agent to reason about the knowledge of another agent. It is the logic of Boolean possibility theory.

Then we recall past works showing that three-valued logics, where the third intermediary truth-value means "unknown", can be expressed in MEL, but the modalities only apply to literals. In particular, Kleene logic corresponds to an even more restricted sublanguage of MEL, where only conjunctions and disjunction of boxed literals are admitted. This embedding of three-valued logics of partial ignorance into MEL enables to better interpret the meaning of three-valued connectives, especially three-valued implications. For instance, Kleene implication corresponds to $\Box\neg p \vee \Box q$, Nelson implication to $\Box p \to \Box q$, and Łukasiewicz one to $(\Box p \to \Box q) \wedge (\Box\neg q \to \Box\neg p)$.

Finally, we discuss the information contained in a formal context (of Formal Concept Analysis) and the attribute implications one extracts from it, in the light of MEL and three-valued logics. See [6] for a unified view of FCA. A formal context is a Boolean matrix containing crosses and blank cells, where a cross indicates that one object is known to possess an attribute. There is an ambiguity in the literature concerning the meaning of blank cells. It ranges from "the object does not possess the attribute" to "it is not known whether the object possesses the attribute or not". Knowledge is extracted from contexts under the form of attribute implications $A \Rightarrow B$ expressing the fact that "if an object possesses all attributes in A it also possesses all attributes in B". Attribute implications can be encoded using conjunction and implication connectives obeying so-called Armstrong axioms. It is a fragment of propositional logic that enables attribute implications valid in the context to be inferred from other ones (see [6] for an overview).

The fact that attribute implications involve only positive attributes suggest that blank cells in contexts are weaker than explicit negation of attributes, and rather express ignorance; so, contexts express partial knowledge. A full-fledged imprecise context will then contain three possible kinds of cells corresponding to positive information, negative information, and ignorance; see [3] for a possibility-theoretic view of incomplete contexts. It also suggests using three-valued logics could encode (generalized) attribute implications, as suggested already in the literature. More generally, we may use the MEL language. A possible translation of a standard attribute implication into MEL is of the form $(\wedge_{a\in A}\Box a) \to (\wedge_{b\in B}\Box b)$, which uses Nelson implication, not Kleene one. It is clear that Armstrong axioms are valid in this translation, and moreover this translation respects the fact that an attribute implication is not considered contrapositive. We conjecture that the logic of hybrid attribute implications (involving both positive and negative literals) extracted from a general imprecise concepts can be expressed in MEL, and that Nelson logic is a good candidate for a three-valued logic that can account for it.

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